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Who I am

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Math as I like it /6

From harps to rational points (and numbers)

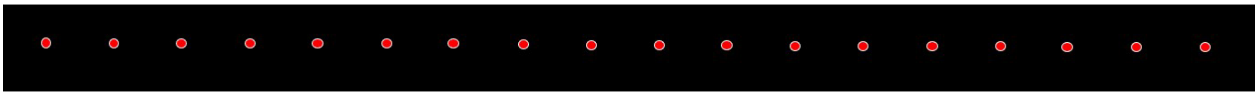
Like the previous one (/5), this episode is very loosely based on the article "Les harpes de Thalès" ("Thales' harps") that I published on the "Images des mathématiques (CNRS)" website: ***a geometric approach to numbers****. Very loosely because here I'm not writing about math,* ***I'm telling it… the way I like it*** *. And as I'd like to share it with you. But of course, fundamentally, it's the same math!*

Do you remember the surveyor robot from the previous episode (/5 What if numbers were just names?)? The distant cousin of the lamplighter from The Little Prince who, like him, lights up the lamps without asking questions, because it's the instructions. Except that it, of course, doesn't constantly turn the same streetlight on and off (day, night… day, night…), it spends its eternity lighting up different points on the same line. Points that I've called "whole points."

Oh no, don't start quibbling; we both know very well that you can't "light up" a point. Because a point is just a place: the surveyor "lights up" a point by injecting it with a bright point object.

(I might be a little grumpy these days – it must be my age… so please forgive me if I'm getting worked up over nothing: maybe you *really* haven't read Episode 5. In that case, please read it – this episode is a sequel to it!)

Anyway, starting from two points in space that I had kindly lit up for it, it had in turn lit up an infinite number of others – all of them points on the line containing the first two. The result looked something like this:



(Imagine them against the black background of empty space: a vast ribbon of stars in your geometric universe!)

And to each of these points we assigned an identity card, a "number" that carries within it a logical construction allowing us to identify the point based on the first two points I had lit up.

But these lit points are still just a few points on the line I chose (by lighting up two points): its "whole points."

How can we light up other points on this line – other than randomly, which would only allow us to identify them by pointing to them?

Let the surveyor robot continue to light up its whole points (it has time; it has eternity on its side). For our part, we must now think more subtly, inventing a mechanism that will in turn light up many new points… and whose logic will allow us to associate a new number with each of them – its identity card.
And this identity card must obviously be based on the mechanism used to reach that point, so that we can still find it again, identify it from the two initial points of our gradation.

This mechanism is a *theorem*! No, don't leave just yet, don't run away! That word has practically become a dirty word (to the point that it appears only twice, I believe, in official French middle school curricula) and I wouldn't want to shock you with my vocabulary… but would you mind making a tiny exception for this particular theorem? You'll see, it's worth it – and besides, as I told you, here I don't "do" math, I tell it!

So I'm going to *tell* you about it! And to start, I'm going to give it a name because, really, it deserves one (math is full of nameless theorems – theorems we have to describe or quote in full every time we need to use them, and that's… that's a bit insulting to them, isn't it?)

Why does it "deserve" a name? Because a whole branch of the study of numbers rests on it – and also because it's incredibly simple. I've called it the "***Thales' harps theorem***".

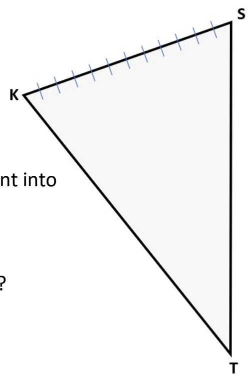
Why? And what's it *talking about*?

Let's start with what it's about. Okay, here I'm going to cheat a little bit and talk about the "length" of a line segment: strictly speaking, I'm not supposed to do that because the concept of length relies on numbers… and we're just starting to invent them; we only know whole numbers (or "integers") so far! But since here I'm *telling* the theorem, I don't really need to be rigorous – the important thing is that we understand each other!

(Quick reminder: if you want *truly* rigorous math, it's here: [*Thales' harps*]() ... in French 😊)

So, the **principle** of *Thales' harps theorem*:

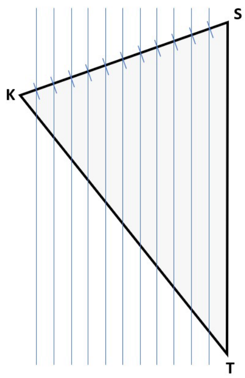
Take a triangle, any triangle,
for example this one (its vertices are points K, S, and T),
and you divide one of its sides – for example, [KS] –
into several segments that are all the same length:



But *how* do you divide a segment into several segments of the same length?

Okay, I'*m just telling* here, okay?

Then you choose a second side – for example, [ST] –
and imagine the lines parallel to (ST) that pass
through the endpoints of these segments:



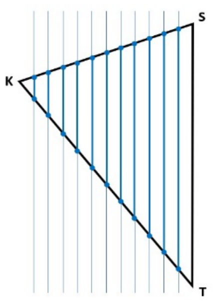
Well, yes, the theorem states that these lines divide the third side – that is, [KT] – into segments that are all the same length (but generally not the same length as the small segments of [KS])! You probably guessed that!

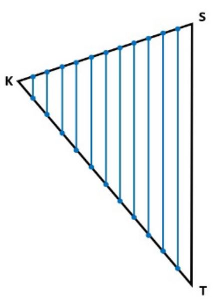
Is that a theorem? Yes, of course, and a really useful one!

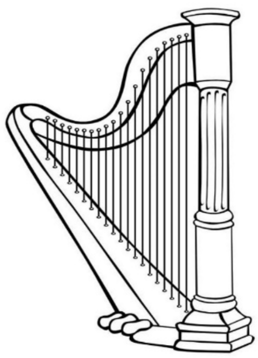
A theorem, after all, is simply a statement that we've managed to prove is true.
And the beauty of this one is that its proof is really, really, really simple (I won't write it here, but you can find it in "[*Les harpes de Thalès*]()", it's note 4, the last one, right at the bottom of the article on [*Images des mathématiques*]())!

And now, why *"the **harps*** theorem"?

It's obviously very personal…
But if, out of all the lines parallel to **(ST)**, you keep only the line segments that pass through the triangle, does that really seem so unreasonable to you?







😊 **Do you believe it?** 😊

I had thought that ***"Thales' harps"*** would make a nice title for my article on "Images des Mathematiques"… an article *about math*!

Well, I discovered that at the same time (*give or take* 50 years) and in the same place (*give or take* 200 km), there were **two Thales**:
Thales of Miletus – the one associated with math and the theorem (and, quite incidentally, with my article)
… and **Thales of Crete** (or "of Candia") – a poet and musician, ***who was summoned to Sparta to ward off the plague by playing the harp there!***

What? “Theorem,” sure, “harps,” sure… but why “of Thales”? What does it have to do with Thales’ theorem, aside from the fact that there are parallels in a triangle?

Well, it has *everything* to do with it! In fact, it foreshadows Thales’ theorem, it’s a primitive form of it. Why primitive? Because it is restricted to a countable set… but “restricted” is a bit of a strong word: this elementary theorem will still allow us to light up a huge number of new points on the line (an infinity of infinities) – and to assign a “logical” number to each of them!

The harps of light:

How am I going to make use of this “harps theorem”?

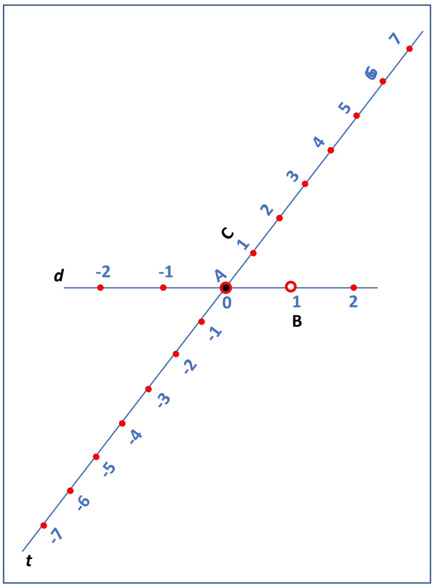
I start from points **A** and **B** in [the previous episode],
the line ***d*** that passes through them, and the ***whole gradation***
– with ***origin*** **A** and ***unit point*** **B** – that they define on this line,

I call “elementary segments” of ***d*** those segments that have as endpoints two consecutive whole points on ***d***
(two whole points, one of which is the successor of the other).

Next, I choose a point that does not lie on ***d***,
I call it **C,** and I call ***t*** the line passing through **A** and **C**.

And finally, I call upon a new surveyor robot
to light up *on line* ***t*** the whole points of the gradation
that has **A** as its origin and **C** as its unit point

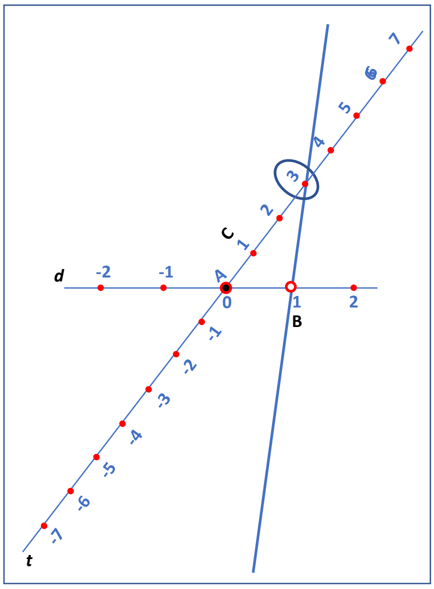
(Oh, and I write down the ***abscissas*** of the whole points on both gradations!)



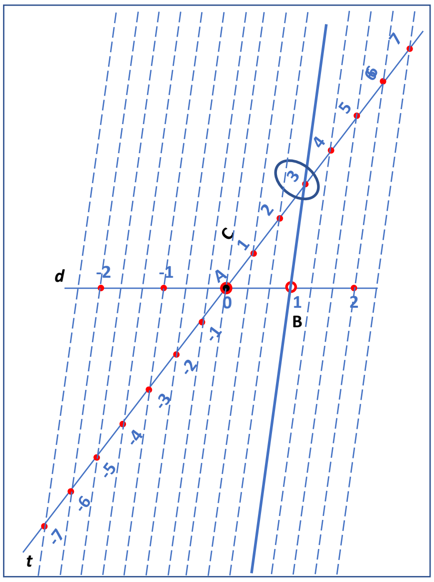
And it’s (almost) done:

would you like to light up points that will divide *each* of the elementary segments of ***d*** into **3** segments that are “three times smaller” (3 “*congruent*” segments 😊)?

All you have to do is imagine the line that passes through **B** and the point on the line ***t*** with an abscissa of 3…



then imagine all the lines parallel to it
and that pass through the other whole points on ***t***:



The harp theorem does the rest 😊!

Not quite clear?

Okay, then what do you think of this:

now imagine keeping only the portions of these parallel lines between ***d*** and ***t*.**

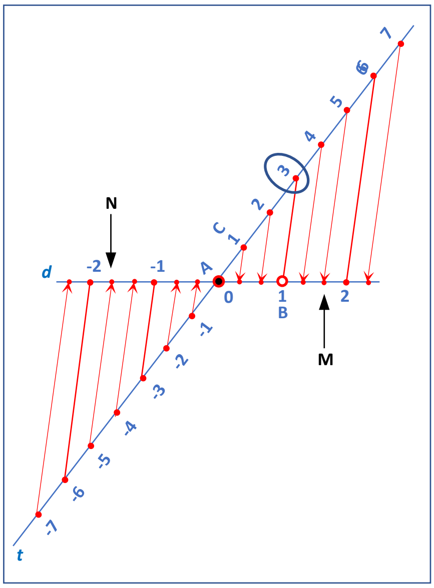
Imagine further that these portions are infinitely thin optical fibers that transmit light from the whole points on ***t*** to points on ***d***:

some of these points (the whole points on ***d***) were *already* lit, but others, such as **M** and **N,** were not… and now they are!

And this is where the harps theorem comes into play:

since the elementary segments of ***t*** all have the same length, the segments that the parallel lines subdivide on ***d*** also all have the same length (yes, generally a *different* length!)

And so we have a lighting up of points on ***d*** **3** times finer than the previous one!



Here's where it all comes together: if you prefer a subdivision of ***d*** that is 317 times finer … just imagine a harp whose optical fibers (the strings) are parallel to the line passing through **B** and the point of ***t*** with an abscissa of **317**!

Isn't that extraordinary?

All that's left for us to do is to come up with and assign a "logical" name (a number) to each of the new lit points.

Which one? Well, for example, for **M**:

the point with abscissa **5** *of **t*** transmits its light to M via a fiber of "harp **3**" (whose strings are parallel to the line passing through **B** and the point with abscissa **3** *of **t***), *so why not "lit up by **5** via harp **3**"* … or more simply **5/3**!
(*Since **N** is the symmetric point of **M** with respect to the origin of **d**, its name (number) will be **-5/3***). Okay, fine, it's just a story😊 😊 😊 !

We can obviously create harps of light whose strings are as tight as we wish: each of the billions of positive integer points of ***t*** (with the exception of **A**) generates a harp whose billions of strings are less tight (and less slanted) than those of the harp generated by the next positive integer point.

Can you picture these billions of harps with billions of strings that are more or less slanted, more or less tight? As for me, no, I can't!

As we reach the end of this episode, perhaps it's time to point out that the numbers associated with the billions upon billions of points lit up by these harps of light are called "**rational numbers**"! And I suppose it won't surprise you that I call the corresponding extension of our "integer (or whole) gradation of d" the "**rational gradation** of ***d*** "… and the points on this gradation the "**rational points** of ***d*** "?

(Of course, the "integer points" of ***d*** are part of these "rational points" since each of them is also targeted *(unnecessarily since it's already lit!)* by one of the integer points of ***t***. But you surely know that integers are special kinds of rational numbers!)

Have we finished lighting up the line ***d***?

No, far from it (and you know that, of course), but as Kipling would say, that is another story: the points and irrational numbers will appear in due time… and that time is still far off (I don't mean to be rude to them, but for now we can certainly do without them)!

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